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Ex Compute volume of the disk of radius α (solid ball)

Note: We already did this in cartesian coordinates and it was nasty

sol/ In spherical coordinates, $D_\alpha = \{(\rho, \theta, \phi) : 0 \le \rho \le \alpha \quad 0 \le \theta \le 2\pi \quad 0 \le \phi \le \pi\}$

$$\mathrm{Vol}(D_\alpha) = \iiint_{D_\alpha} 1 \, dV_{cart} \xrightarrow{\text{spherical}} \iiint_{D_\alpha} 1 \, \rho^2 \sin(\phi) \, dV_{spherical}$$

$$\int_{0=\rho}^{\alpha} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \rho^2 \sin(\phi) \, d\phi \, d\theta \, d\rho$$

$$= \int_0^{\rho=\alpha} \int_0^{\theta=2\pi} -\rho^2 \cos(\phi) \Big]_0^{\pi} = -\rho^2(-1-1)$$

$$2 \int_{\rho=0}^{\alpha} \int_0^{2\pi=\theta} \rho^2 = \int_0^{\rho=\alpha} \rho^2 \theta \Big]_0^{2\pi} = 2\pi \rho^2$$

$$4\pi \int_{\rho=0}^{\alpha} \pi \rho^2 = \frac{4\pi\rho^3}{3} \Big]_0^{\alpha} = \frac{4\pi\alpha^3}{3} - 0$$

$$= \frac{4\pi\alpha^3}{3}$$

16.2 Vector fields

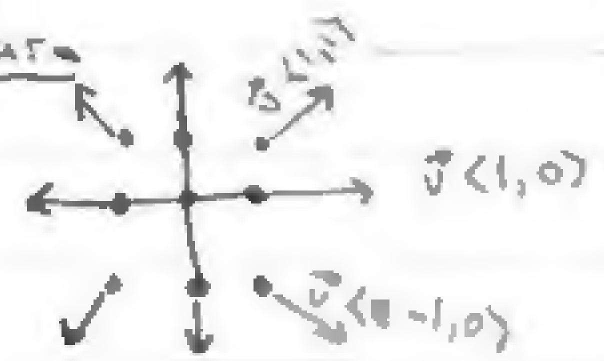
Goal: Study $\vec{v}: \mathbb{R}^n \to \mathbb{R}^n$

$\boxed{\text{"vf" = vector field}}$

Def: A vector field on $\mathbb{R}^n$ is a function $\vec{v}: \mathbb{R}^n \to \mathbb{R}^n$

Ex $\vec{v}(x,y) = \langle x,y \rangle$ is a v.f. on $\mathbb{R}^2$

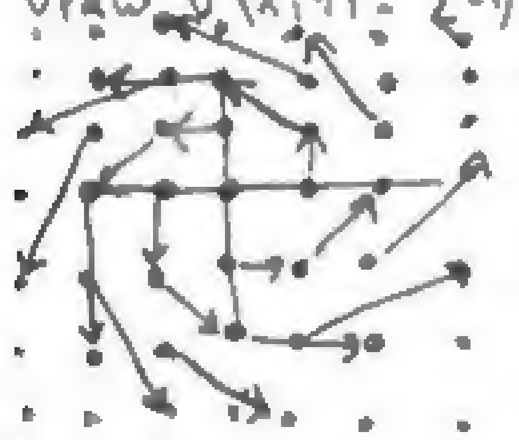
picture



We shift the vector $\vec{v}\langle x,y\rangle$ to have tail (x,y)

Ex

Draw $\vec{v}(x,y) = \langle -y, x\rangle$



Think of it like a hurricane where $(0,0)$ is an eye of the storm

Vector fields are like mapping a force

Can describe how a force interacts

$\vec{v}(0,0) = \langle 0,0\rangle$

~~$\vec{v}(1,0) = \langle 0,1\rangle$~~ $\vec{v}(1,0) = \langle 0,1\rangle$

~~don~~ $v(0,1) = \langle -1,0\rangle$

$v(1,1) = \langle -1,1\rangle$ (etc.)

$v(1,2) = \langle -2,1\rangle$

$v(2,1) = \langle -1,2\rangle$

Ex

Given any function $f: \mathbb{R}^n \to \mathbb{R}$, we obtain a vector field by taking the gradient:

e.g. $f(x,y) = xy$

$\nabla f = \langle y, x \rangle$ is the gradient vector field of f

e.g. $f(x,y,z) = e^{x+y^2}\cos(x+z)$

$\nabla f(x,y,z) = \langle e^{x+y^2}\cos(x+z) - e^{x+y^2}\sin(x+z),\ 2y e^{x+y^2}\cos(x+z),\ -e^{x+y^2}\sin(x+z)\rangle$

is a v.f. on $\mathbb{R}^3$

e.g. $f(x,y) = x^3 + 3xy - xy^2$

$\nabla f = (3x^2 + 3y - y^2,\ 3x - 2xy)$

Terminology: ① A vector field is conservative when it is the gradient v.f. of some f

② When $\vec{v} = \nabla f$ is conservative we say f is a potential function for $\vec{v}$

Obvious Question: Which v.f.s are conservative?

↳ "aren't all of them conservative"

If $\vec{v}(x,y)$ is conservative, then $\vec{v} = \nabla \vec{f}(x,y)$

i.e. $\vec{v}(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

By Clairaut's Theorem, $f_{xy} = f_{yx}$

So for $\vec{v} = \langle v_x, v_y \rangle$ we have

$$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial x}[v_y] \text{ for all conservative v.f.s}$$

Can construct non-conservative v.f.s easily

$\vec{v} = \langle -y, x \rangle$ is <u>not</u> conservative b/c $\left.\begin{array}{l} \frac{\partial}{\partial x}[v_y] = 1 \\ \frac{\partial}{\partial y}[v_x] = -1 \end{array}\right\}$ Are not equal

It turns out this is an "iff" type condition!

prop: A vector field $\vec{v}(x_1, x_2, \dots, x_n) = \langle v_1, v_2, \dots, v_n \rangle$ is conservative if and only if for all i, j we have:

$$\frac{\partial}{\partial x_i}[v_j] = \frac{\partial}{\partial x_j}[v_i]$$

(i.e. A vector is only conservative iff it satisfies Clairaut's Theorem)

Note: A proof of this result follows from the methods I give below...

Ex

Is $\vec{v} = \langle x, y \rangle$ conservative? If yes, potential?

Sol

$$\frac{\partial}{\partial x}[v_y] = \frac{\partial}{\partial x}[y] = 0$$

$$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial y}[x] = 0$$

To compute the potential:

If $\vec{v} = \nabla f$, then $f_x(x,y) = x \quad f_y(x,y) = y$

$\therefore f(x,y) = \int \frac{\partial f}{\partial x}\, dx = \int x\, dx = \frac{1}{2}x^2 + C(y)$ Must not depend on x, as if it was the derivative couldn't be 0

$\frac{1}{2}x^2 + C(y)$

Find $C(y)$

$\therefore y = f_y(x,y) = \frac{\partial}{\partial y}\left[\frac{1}{2}x^2 + C(y)\right] = y$

hence: $C(y) = \int y\, dy$ an actual constant

$C(y) = \frac{1}{2}y^2 + D \qquad f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$

continued

$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$ is a potential for $\vec{v}$ for every constant D

Ex

Is $\vec{v} = \langle 2xy, x^2 - 3y^2 \rangle$ conservative? If yes, find its potential

sol $\frac{d}{dx}[V_y] = 2x \quad \frac{d}{dy}[V_x] = 2x$

$2x = 2x$, V is cons

find its potential

$\vec{V} = \nabla f$ for some $f(x,y)$ where

~~$\frac{d}{dx}[V_y] = 2x \quad \frac{d}{dy}[V_x] = 2x$~~

$f_x(x,y) = 2xy \quad f_y(x,y) = x^2 - 3y^2$

$\therefore f(x,y) = \int \frac{df}{dx}\,dx = \int 2xy\,dx = \frac{2x^2y}{2} + C(y)$

Hence $x^2 - 3y^2 = \frac{df}{dy} = \frac{d}{dy}[x^2y + C(y)]$

$\cancel{x^2} - 3y^2 = \cancel{x^2} + \frac{dc}{dy}$

$\int \frac{dc}{dy} = \int -3y^2$

$= -\frac{3y^3}{3} + D$

$C(y) = -y^3 + D$

Potential is $x^2y - y^3 + D$ for every constant D